ISSN: 0974-2115
www.jchps.com Journal of Chemical and Pharmaceutical Sciences

Oscillatory and Non Oscillatory Properties of Functional Difference Equations and Its Applications

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ABSTRACT

The aim of this paper is to study the oscillatory and non-oscillatory properties of a functional difference equation of the form $\Delta^2_\alpha u(k) = f(k, u(k), \Delta_\beta u(k))$, $k \in N$ ...(1) Where α, β are real fixed constants, $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$, $\Delta^2_\alpha u(k) = \Delta_\alpha[\Delta_\alpha u(k)]$, $\Delta_\beta u(k) = u(k+1) - \beta u(k)$, and f is defined on N xR^2

KEY WORDS: difference equations, oscillation, non-oscillation, Mathematics Subject Classification (2010): 39A10, 33A21

1. INTRODUCTION

Difference equations arise in the situations in which the discrete values of the independent variable are involved; many practical phenomena are modeled with the help of difference equations. In engineering, difference equations arise in control engineering, digital signal processing, electrical networks, etc. In social sciences, difference equations are used to study the national income of a country and then its variation with time, Cobweb phenomenon in economics, etc. Analogous to differential equation, difference equation is the most powerful instrument for the treatment of discrete processes.

In this paper we will study the oscillatory and non-oscillatory properties of some Functional difference equation

2. MATERIALS AND METHODS

Using the methods of Oscillation and non-oscillation we briefly explained all the Theorems.

Definition 1: A difference equation is an equation which expresses a relation between an independent variable and the successive values of the dependent variable or the successive differences of the dependent variable.
Example,

$$y_{k+n} = F[k, y_{k+n-1}, y_{k+n-2}, \ldots y_k]$$

(n^{th} order Difference Equation)

Definition 2: Order of a Difference Equation: The difference between the largest and smallest arguments appearing in the difference equation is called its order. $y_{k+1} - 3y_k + y_{k-1} = e^{-k}$ (2^{nd} order Difference Equation)

Definition 3: Solution of a Difference Equation: A solution of a difference equation is a relation between the independent variable and the dependent variable satisfying the equation. For example, the function $\phi(k) = 2^k$ is a solution of the first order linear difference equation $y_{k+1} - 2y_k = 0$. Since on substitution of $\phi(k)$ into the equation an identity is obtained in $2^{k+1} - 2.2^k = 0$.

Definition 4: Oscillation: Let $y = y(n) = (\{y_n\})_{n=1}^{\infty}$, $n \in N$, y is a real valued sequence. For this y the difference operators Δ, Δ_a, $a \in R$. And $\Delta^k, k \in N$ are defined as follows.

$$\Delta y_n = y_{n+1} - y_n \quad , \Delta_a y_n = y_{n+1} - a y_n \quad , n \in N$$

Definition 5: The sequence y is said to be oscillatory around a ($a \in R$) if there exists an increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that $(y_{n_k} - a)(y_{n_{k+1}} - a) \le 0 \quad for\ all\ k \in N$.

3. RESULTS AND DISCUSSION

Theorem 1: Let $\alpha > 0$ and $T = N$ x $\{(u,v) \in R^2$: v+ (β-α) u =0}. Further, let

i. F(k ,u , v) = 0, if (k ,u ,v) ∈ T
ii. F(k , u, v) [v + (β-α) u + α[v + (β-α) u]$^2 > 0$, if (k , u , v) ∈ N x R^2/ T

Then, the difference equation (1) is non-oscillatory.

Proof: We observe that (k, u (k), $\Delta_\beta u(k) \in T$ is equivalent to u (k+1) – α u (k) = 0. Therefore, if the solution u (k) of (1) is such that for a fixed $k_1 \in T, (k_1, u(k_1), \Delta_\beta u(k_1)) \in T$, then from the hypothesis (i), it follows that $\Delta^2_\alpha u(k_1) = 0$. However, since $\Delta^2_\alpha u(k_1) = \Delta_\alpha u(k_1+1) - \alpha\Delta_\alpha u(k_1) = u(k_1+2) - \alpha u(k_1+1) = 0$ (2)
Inductively, we have $u(k_1+l) - \alpha u(k_1+l-1) = 0,\ l \in N(1)$, and hence $u(l) = \alpha^{l-k_1} u(k_1), l \in N(k_1)$. This solution is of course non oscillatory.

Now let u(k) be a solution of (1) such that for any $k \in T, (k, u(k), \Delta_\beta u(k)) \notin T$, and this solution is oscillatory. Then, there exists a $k_2 \in N$ such that $u(k2)>0, u(k2+1)\le 0$ and hence $\Delta_\alpha u(k_2)<0$. Setting $k=k_2$ in (1) and multiplying the resulting equation by $\Delta_\alpha u(k_2)$gives

$$\Delta_\alpha u(k_2)\Delta_\alpha u(k_2+1) = f(k_2, u(k_2), \Delta_\beta u(k_2))[\Delta_\beta u(k_2)+(\beta-\alpha)u(k_2)] +$$
$$\alpha[\Delta_\beta u(k_2)+(\beta-\alpha)u(k_2)]+\alpha[\Delta_\beta u(k_2)+(\beta-\alpha)u(k_2)]^2$$

Therefore, from the hypothesis (ii), we find that $\Delta_\alpha u(k_2)\Delta_\alpha u(k_2+1)>0,$ *and hence* $\Delta_\alpha u(k_2+1)<0.$ repeating this reasoning we get $\Delta_\alpha u(k)<0$ *for all* $k \in N(k_2)$. This implies that $u(k)<0$ *for all* $k \in N(k_2+2)$. This contradicts our assumption. The proof for the case $u(k_2)\ge 0$, $u(k_2+1)<0$ is similar.

Theorem 2: Let $\alpha>0$ and $S=N$ x$\{(u,v) \in R^2: v+\beta u=0)\}$ Further, let $F (k, u, v) (v+\beta u) \alpha (v+\beta u) [v + (\beta-\alpha) u] > 0$ if $(k, u, v) \in N \, x \, R^2 \setminus S$.

Then, the difference equation (1) is non-oscillatory.

Proof: We observe that $(k, u(k), \Delta_\beta u(k) \in S$ is equivalent to $u(k+1) = 0$. Since we consider only nontrivial solution, there exists a $k_1 \in N$ such that u (k$u(k_1+1)) = \Delta_\beta u(k_1)+\beta u(k_1) \ne 0.$

Setting $k=k_1$ in (1) and multiplying the resulting equation by $u(k_1+1)$ gives

$$u(k_1+1)\Delta_\alpha u(k_1+1) =$$
$$f(k_1, u(k_1), \Delta_\beta u(k_1))u(k_1+1)+\alpha u(k_1+1)\Delta_\alpha u(k_1)$$
$$= f(k_1, u(k_1), \Delta_\beta u(k_1))(\Delta_\beta u(k_1)+\beta u(k_1))$$
$$+ \alpha(\Delta_\beta u(k_1)+\beta u(k_1))[\Delta_\beta u(k_1)+(\beta-\alpha)u(k_1)] \quad (3)$$

Hence, from the given hypothesis it follows that $u(k_1+1)\Delta_\alpha u(k_1+1)>0.$ then $\Delta_\alpha u(k_1+1)>0$ implies $u(k_1+2)>\alpha u(k_1+1)>0$

Repeating the above reasoning we obtain $\Delta_\alpha u(k)>0$ *for all* $k \in N(k_1+1)$, and from this $u(k)>\alpha^{k-k_1-1}u(k_1+1)>0$ for all $k \in N(k_1+2)$. This solution is positive and therefore non oscillatory. A similar proof holds for $u(k_1+1)<0$.

Theorem 3: Let $\alpha=\beta=1$and $f(k, u, v) (u+v) \ge 0$ if $(k, u, v) \in N$ x R^2.Then, the difference equation (1) is non-oscillatory.

Proof: Suppose that there exists an oscillatory solution of $u(k)$ of (1) . Then, there exists $k1,k2 \in N$ such that $u(k_1)\le 0$, $u(k_2\ge 0$, $N(k_1+1, k_2-1)$ is non-empty and finite, and there exists a $l \in N(k1+1,k2-1)$ such that $u(l)>0$ and simultaneously $u(l)>u(l+1), u(l)\ge u(l-1)$. Thus $\Delta^2 u(l-1)=\Delta u(l)-\Delta u(l-1)<0$

But, setting $k=l-1$ in (1) and multiplying the resulting equation by $u(l)$ gives

$u(l)\Delta^2 u(l-1) = u(l) f(l-1, u(l-1), \Delta u(l-1))$ And hence, from the given hypothesis we have $u(l)\Delta^2 u(l-1)>0$, (i. e) $\Delta^2 u(l-1)>0$. This contradiction completes the proof.

Theorem 4: Let in theorem 1 the inequality sign '> 'be replaced by '< 'at both the places. Then the difference equation (1) is oscillatory.

Theorem 5: Let in theorem 2 the inequality sign '> 'be replaced by '< 'at both the places. Then the difference equation (1) is oscillatory.

Example: $u(k+2)-(1+u^2(k))u(k+1)+u^3(k)$, $k \in N$

By rewriting this equation in the form $\Delta_\alpha^2 u(k) = f(k, u(k), \Delta_\beta u(k))$, $k \in N$ where $f(k,u,v) = u^2 v - v$

We see that the condition (i) theorem 1 with $\alpha=1, \beta=1$ is satisfied. Hence all nontrivial solutions of this equation are non-oscillatory.

Example for difference equation:

Let $u(k)=u(k-1)+u(k-2)$…… (i)

Equation (i) defines then so-called Fibonacci sequence. Starting with $u(0)=1$ *and* $u(1)=1$ We can easily calculate each following terms of the sequence: 1, 1, 2, 3, 5, 8, 13, 21, 35 ... (ii) Note that each term can be computed only if the two first terms of the sequence are given. Those terms are called the initial conditions of the system.

Table.1 Difference table (Fibonacci numbers)

k	0	1	2	3	4	5	6	….
u(k)	1	1	2	3	5	8	13	….

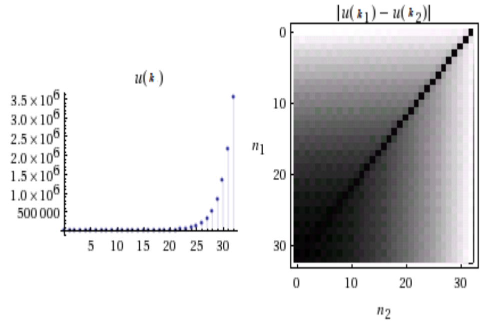

Figure.1. Graph of the Fibonacci sequence

Difference equations have many applications in population dynamics. For example, Eq. (i) can be seen as a (very simple) model for growth and reproduction of a rabbit population.

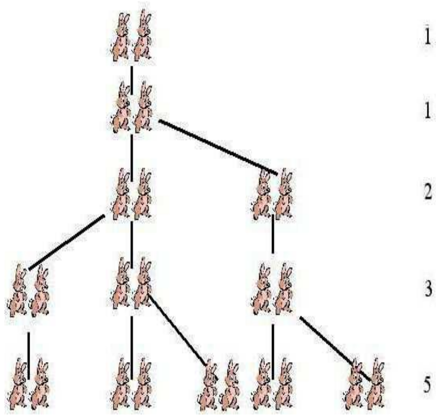

Figure.2. Growth of population of rabbits (Fibonacci sequence).
Ref: Linear Difference Equations by Didier Gonze

4. CONCLUSION

Based on Theorems 1, 2 and 3 the equation (1) is Non oscillatory and by theorems 4 and 5 the equation (1) is Oscillatory.

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